

A Lower Bound on the Relative Entropy with Respect to a Symmetric Probability

Raphaël Cerf
ENS Paris

and

Matthias Gorny
Université Paris Sud and ENS Paris

Abstract

Let ρ and μ be two probability measures on \mathbb{R} which are not the Dirac mass at 0. We denote by $H(\mu|\rho)$ the relative entropy of μ with respect to ρ . We prove that, if ρ is symmetric and μ has a finite first moment, then

$$H(\mu|\rho) \geq \frac{\left(\int_{\mathbb{R}} z d\mu(z)\right)^2}{2 \int_{\mathbb{R}} z^2 d\mu(z)},$$

with equality if and only if $\mu = \rho$.

AMS 2010 subject classifications: 60E15 94A17

Keywords: entropy, Cramér transform

1 Introduction

Given two probability measures μ and ρ on \mathbb{R} , the relative entropy of μ with respect to ρ (or the Kullback-Leibler divergence of ρ from μ) is

$$H(\mu|\rho) = \begin{cases} \int_{\mathbb{R}} f(z) \ln f(z) d\rho(z) & \text{if } \mu \ll \rho \text{ and } f = \frac{d\mu}{d\rho} \\ +\infty & \text{otherwise,} \end{cases}$$

where $d\mu/d\rho$ denotes the Radon-Nikodym derivative of μ with respect to ρ when it exists. In this paper, we prove the following theorem:

Theorem. *Let ρ and μ be two probability measures on \mathbb{R} which are not the Dirac mass at 0. We suppose that*

$$\int_{\mathbb{R}} |z| d\mu(z) < +\infty.$$

If ρ is symmetric then

$$H(\mu|\rho) \geq \frac{\left(\int_{\mathbb{R}} z d\mu(z) \right)^2}{2 \int_{\mathbb{R}} z^2 d\mu(z)},$$

with equality if and only if $\mu = \rho$.

A remarkable feature of this inequality is that the lower bound does not depend on the symmetric probability measure ρ . We found the following related inequality in the literature (see lemma 3.10 of [1]): if ρ is a probability measure on \mathbb{R} whose first moment m exists and such that

$$\exists v > 0 \quad \forall \lambda \in \mathbb{R} \quad \int_{\mathbb{R}} \exp(\lambda(z - m)) d\rho(z) \leq \exp\left(\frac{v\lambda^2}{2}\right),$$

then, for any probability measure μ on \mathbb{R} having a first moment, we have

$$H(\mu|\rho) \geq \frac{1}{2v} \left(\int_{\mathbb{R}} z d\mu(z) - m \right)^2.$$

Our inequality does not require an integrability condition. Instead we assume that ρ is symmetric.

The proof of the theorem is given in the following section. It consists in relating the relative entropy $H(\cdot|\rho)$ and the Cramér transform I of (Z, Z^2) when Z is a random variable with distribution ρ . We then use an inequality on I which we proved initially in [2]. We give here a simplified proof of this inequality.

2 Proof of the theorem

Let ρ and μ be two probability measures on \mathbb{R} which are not the Dirac mass at 0. We first recall that $H(\mu|\rho) \geq 0$, with equality if and only if $\mu = \rho$.

We assume that ρ is symmetric and that μ has a finite first moment. We denote

$$\mathcal{F}(\mu) = \frac{\left(\int_{\mathbb{R}} z d\mu(z) \right)^2}{2 \int_{\mathbb{R}} z^2 d\mu(z)}.$$

If $\mu = \rho$ then $\mathcal{F}(\mu) = 0 = H(\mu|\rho)$. From now onwards we suppose that $\mu \neq \rho$. If the first moment of μ vanishes or if its second moment is infinite, then $\mathcal{F}(\mu) = 0 < H(\mu|\rho)$. Finally, if μ is such that $H(\mu|\rho) = +\infty$, then Jensen's inequality implies that

$$\mathcal{F}(\mu) \leq 1/2 < H(\mu|\rho).$$

In the following, we suppose that

$$\int_{\mathbb{R}} z d\mu(z) \neq 0, \quad \int_{\mathbb{R}} z^2 d\mu(z) < +\infty,$$

and that $H(\mu|\rho) < +\infty$. This implies that $\mu \ll \rho$ and we set $f = d\mu/d\rho$. It follows from Jensen's inequality that, for any μ -integrable function Φ ,

$$\int_{\mathbb{R}} \Phi d\mu - H(\mu|\rho) = \int_{\mathbb{R}} \ln \left(\frac{e^\Phi}{f} \right) d\mu \leq \ln \int_{\mathbb{R}} \frac{e^\Phi}{f} d\mu = \ln \int_{\mathbb{R}} e^\Phi d\rho.$$

As a consequence

$$\sup_{\Phi \in L^1(\mu)} \left\{ \int_{\mathbb{R}} \Phi d\mu - \ln \int_{\mathbb{R}} e^\Phi d\rho \right\} \leq H(\mu|\rho).$$

In order to make appear the first and second moments of ρ , we consider functions Φ of the form $z \mapsto uz + vz^2$, $(u, v) \in \mathbb{R}^2$. This way we obtain

$$I \left(\int_{\mathbb{R}} z d\mu(z), \int_{\mathbb{R}} z^2 d\mu(z) \right) \leq H(\mu|\rho),$$

where

$$\forall (x, y) \in \mathbb{R}^2 \quad I(x, y) = \sup_{(u, v) \in \mathbb{R}^2} \left\{ ux + vy - \ln \int_{\mathbb{R}} e^{uz + vz^2} d\rho(z) \right\}.$$

The function I is the Cramér transform of (Z, Z^2) when Z is a random variable with distribution ρ . In our paper dealing with a Curie-Weiss model of self-organized criticality [2], we proved with the help of the following inequality that, under some integrability condition, the function $(x, y) \mapsto I(x, y) - x^2/(2y)$ has a unique global minimum on $\mathbb{R} \times]0, +\infty[$ at $(0, \int x^2 d\rho(x))$.

Proposition. *If ρ is a symmetric probability measure which is not the Dirac mass at 0, then*

$$\forall x \neq 0 \quad \forall y \neq 0 \quad I(x, y) > \frac{x^2}{2y}.$$

We present here a proof of this proposition which is simpler than in [2].

Proof. Let $x \neq 0$ and $y \neq 0$. By definition of $I(x, y)$, we have

$$\begin{aligned} I(x, y) &\geq x \times \frac{x}{y} + y \times \left(-\frac{x^2}{2y^2} \right) - \ln \int_{\mathbb{R}} \exp \left(\frac{xz}{y} - \frac{x^2 z^2}{2y^2} \right) d\rho(z) \\ &= \frac{x^2}{2y} - \ln \int_{\mathbb{R}} \exp \left(\frac{xz}{y} - \frac{x^2 z^2}{2y^2} \right) d\rho(z). \end{aligned}$$

Let $(s, t) \in \mathbb{R}^2$. By using the symmetry of ρ , we obtain

$$\begin{aligned} \int_{\mathbb{R}} \exp(sz - tz^2) d\rho(z) &= \int_{\mathbb{R}} \exp(-sz - tz^2) d\rho(z) \\ &= \frac{1}{2} \left(\int_{\mathbb{R}} \exp(sz - tz^2) d\rho(z) + \int_{\mathbb{R}} \exp(-sz - tz^2) d\rho(z) \right) \\ &= \int_{\mathbb{R}} \cosh(sz) \exp(-tz^2) d\rho(z). \end{aligned}$$

We choose now $t = s^2/2$. We have the inequality

$$\forall u \in \mathbb{R} \setminus \{0\} \quad \cosh(u) \exp(-u^2/2) < 1.$$

Since ρ is not the Dirac mass at 0, the above inequality implies that

$$\forall s \neq 0 \quad \int_{\mathbb{R}} \cosh(sz) \exp\left(-\frac{s^2 z^2}{2}\right) d\rho(z) < 1.$$

We finally choose $s = x/y$ and we get

$$\int_{\mathbb{R}} \exp \left(\frac{xz}{y} - \frac{x^2 z^2}{2y^2} \right) d\rho(z) < 1.$$

As a consequence

$$I(x, y) \geq \frac{x^2}{2y} - \ln \int_{\mathbb{R}} \exp \left(\frac{xz}{y} - \frac{x^2 z^2}{2y^2} \right) d\rho(z) > \frac{x^2}{2y},$$

which is the desired inequality. \square

By applying the above proposition with

$$x = \int_{\mathbb{R}} z d\mu(z) \neq 0, \quad y = \int_{\mathbb{R}} z^2 d\mu(z) \in]0, +\infty[,$$

we obtain

$$H(\mu|\rho) \geq I \left(\int_{\mathbb{R}} z d\mu(z), \int_{\mathbb{R}} z^2 d\mu(z) \right) > \mathcal{F}(\mu).$$

This ends the proof of theorem 1.

References

- [1] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press, 2013.
- [2] Raphaël Cerf and Matthias Gorny. A Curie-Weiss model of Self-Organized Criticality. *The Annals of Probability*, to appear, 2013.